Imperial College London

# Lecture 4

# Time-domain analysis: Zero-state Response

(Lathi 2.3-2.4.1)

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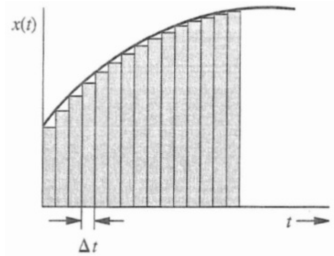
## The importance of Impulse Response h(t)

- **Zero-state response** assumes that the system is in "rest" state, i.e. all internal system variables are zero.
- Deriving and understanding zero-state response depends on knowing the **impulse response** ***h(t)*** to a system.
- Any input $x(t)$ can be broken into many **narrow rectangular pulses**. Each pulse produces a system response.
- Since the system is linear and time invariant, the system response to $x(t)$ is the sum of its responses to all the impulse components.
- $h(t)$ is the system response to the rectangular pulse at $t=0$ as the pulse **width approaches zero**.

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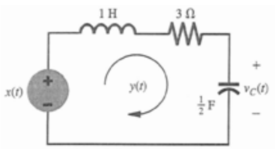
## How to determine the unit impulse response h(t)? (1)

- Given that a system is specified by the following differential equation, determine its unit impulse response h(t).

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y(t)=\frac{dx}{dt}$$




- Remember the general system equation:

$$Q(D)y(t)=P(D)x(t)$$

- It can be shown that the impulse response $h(t)$ is given by:

$$h(t)=[P(D)y_n(t)]u(t) \quad ..... (4.3.1)$$

where $u(t)$ is the unit step function, and **$y_n(t)$ is a linear combination of the** characteristic modes of the system.

$$y_n(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}+........+c_Ne^{\lambda_N t}$$

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## How to determine the unit impulse response h(t)? (2)

- The constants $c_i$ are determined by the following initial conditions:

$$y_n(0)=\dot{y}_n(0)=\ddot{y}_n(0)=....=y_n^{(N-2)}(0)=0, \quad y_n^{(N-1)}(0)=1.$$

- Note $y_n^{(k)}(0)$ is the k^th derivative of $y_n(t)$ at t = 0.
- The above is true if M, the order of P(D), is less than N, the order of Q(D) (which is generally the case for most stable systems).

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## The Example (1)

- Determine the impulse response for the system: $\left(D^2+3D+2\right)y(t)=Dx(t)$
- This is a second-order system (i.e. N=2, M=1) and the characteristic polynomial is:

$$\left(\lambda^2+3\lambda+2\right)=(\lambda+1)(\lambda+2)$$

- The characteristic roots are λ = -1 and λ = -2.
- Therefore : $y_n(t)=c_1e^{-t}+c_2e^{-2t}$
- Differentiating this equation yields: $\dot{y}_n(t)=-c_1e^{-t}-2c_2e^{-2t}$
- The initial conditions are

$$\dot{y}_n(0)=1 \quad \text{and} \quad y_n(0)=0$$

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## The Example (2)

- Setting $t = 0$ and substituting the initial conditions yield:

$$0=c_1+c_2$$
$$1=-c_1-2c_2$$

- The solution of these equations are:

$$c_1=1 \quad \text{and} \quad c_2=-1$$

- Therefore we obtain $y_n(t)=e^{-t}-e^{-2t}$
- Remember that $h(t)$ is given by: $h(t)=[P(D)y_n(t)]\,u(t)$

  and $P(D) = D$ in this case.
- Therefore

$$h(t)=[P(D)y_n(t)]u(t)=(-e^{-t}+2e^{-2t})u(t)$$

$$P(D)y_n(t)=Dy_n(t)=\dot{y}_n(t)=-e^{-t}+2e^{-2t}$$

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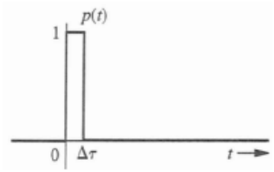
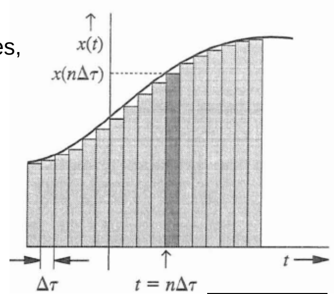
## Zero-state Response (1)

- We now consider how to determine the **system response** $y(t)$ to an input $x(t)$ when system is in zero state.
- Define a pulse $p(t)$ of unit height and width Δτ at t=0:
- Input $x(t)$ can be represented as sum of narrow rectangular pulses.
- The pulse at $t = n\Delta\tau$ has a height $x(t) = x(n\Delta\tau)$.
- This can be expressed as $x(n\Delta\tau)\,p(t - n\Delta\tau)$.
- Therefore $x(t)$ is the sum of all $[x(n\Delta\tau)/\Delta\tau]$. such pulses, i.e.




$$x(t)=\lim_{\Delta\tau\to 0}\sum_{r} x(n\Delta\tau)\,p(t-n\Delta\tau)$$

$$=\lim_{\Delta\tau\to 0}\sum_{r}\left[\frac{x(n\Delta\tau)}{\Delta\tau}\right]p(t-n\Delta\tau)\Delta\tau$$

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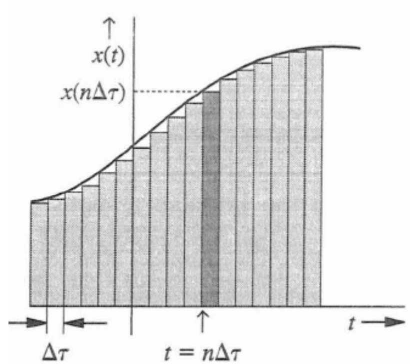
## Zero-state Response (1)

- The term $[x(n\Delta\tau)/\Delta\tau]p(t-n\Delta\tau)$ represents a pulse $p(t - n\Delta\tau)$ with height $x(n\Delta\tau)$
- As Δτ → 0, height of strip → ∞, but area remain x(nΔτ), and

$$\frac{x(n\Delta\tau)}{\Delta\tau}\,p(t-n\Delta\tau)\;\rightarrow\; x(n\Delta\tau)\,\delta(t-n\Delta\tau)$$

$$x(t)=\lim_{\Delta\tau\to 0}\sum_{r} x(n\Delta\tau)\,\delta(t-n\Delta\tau)\,\Delta\tau$$

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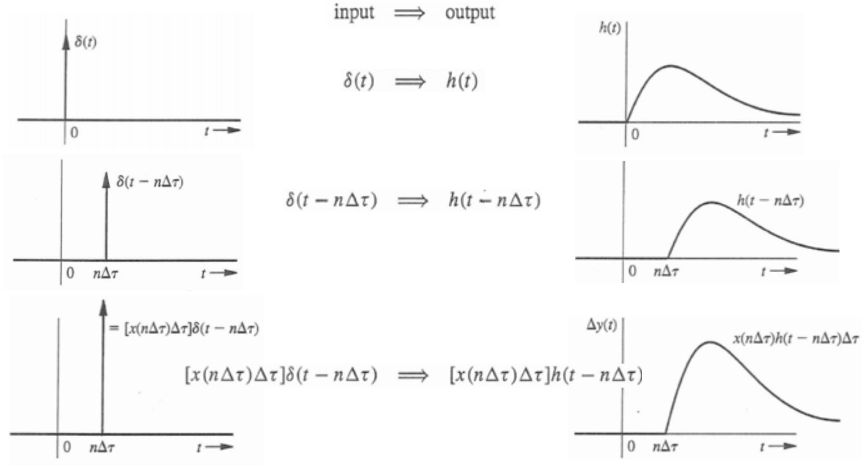
# Zero-state Response (2)

input $\Longrightarrow$ output

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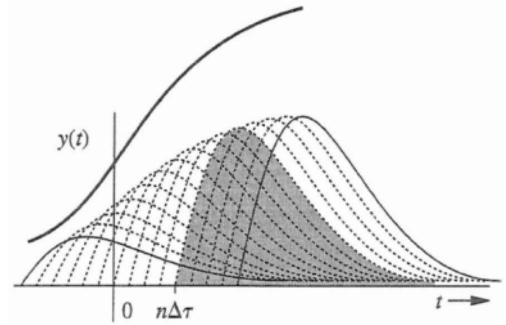
# Zero-state Response (3)

input $\Longrightarrow$ output

$$\underbrace{\lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)\delta(t - n\Delta\tau)\,\Delta\tau}_{x(t)} \Longrightarrow \underbrace{\lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau}_{y(t)}$$

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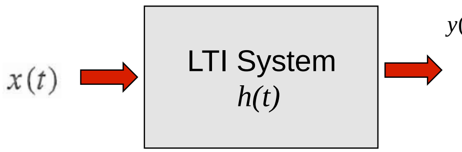
# Zero-state Response (4)

- Therefore,

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\,d\tau$$

- Knowing h(t), we can determine the response y(t) to any input x(t).
- Observe the all-pervasive nature of the system's characteristic modes, which determines the impulse response of the system.




$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\,d\tau$$

L2.4 p171



# The Convolution Integral

- The derived integral equation occurs frequently in physical sciences, engineering and mathematics.
- It is given the name: the convolution integral.
- The convolution integral of two functions $x_1(t)$ and $x_2(t)$ is denoted symbolically as

$$x_1(t) * x_2(t)$$

- And is defined as

$$x_1(t) * x_2(t) \equiv \int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau)\,d\tau$$

- Note that the convolution operator is linear, i.e. it obeys the principle of superposition.

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# Properties of Convolution (1)

- COMMUTATIVE PROPERTY (order of operands does not matter):

$$x_1(t) * x_2(t) = -\int_{\infty}^{-\infty} x_2(z)x_1(t-z)\,dz \qquad \text{let } z = t - \tau$$
$$= \int_{-\infty}^{\infty} x_2(z)x_1(t-z)\,dz$$
$$= x_2(t) * x_1(t)$$

- ASSOCIATIVE PROPERTY (order of operator does not matter):

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

- DISTRIBUTIVE PROPERTY:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

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# Properties of Convolution (2)

- SHIFT PROPERTY:
  - If $\quad x_1(t) * x_2(t) = c(t)$
  - then $\quad x_1(t) * x_2(t-T) = x_1(t-T) * x_2(t) = c(t-T)$
  - Also $\quad x_1(t-T_1) * x_2(t-T_2) = c(t-T_1-T_2)$
- IMPULSE PROPERTY:
  - Convolution of a function $x(t)$ with a unit impulse results in the function $x(t)$.

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \quad = x(t)$$

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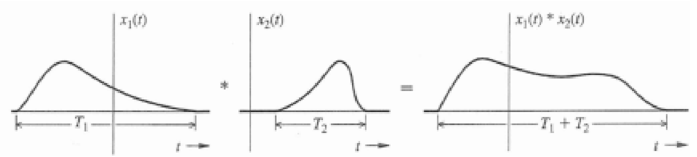
# Properties of Convolution (3)

- WIDTH PROPERTY:
  Duration of $x_1(t) = T_{1,}$ and duration of $x_2(t) = T_{2,}$
  then duration of $x_1(t) * x_2(t) = T_1 + T_{2,}$




- CAUSALITY PROPERTY:
  If both system's impulse response $h(t)$ and the input $x(t)$ are causal, then

$$y(t) = x(t) * h(t) = \int_{0^-}^{t} x(\tau)h(t-\tau)\,d\tau \qquad t \geq 0$$
$$= \int_{0^-}^{t} h(\tau)x(t-\tau)\,d\tau$$
$$= 0 \qquad t < 0$$

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# Example (1)

- For a LTI system with the unit impulse response $h(t) = e^{-2t}u(t)$, determine the response y(t) for the input $x(t) = e^{-t}u(t)$
- Both $h(t)$ and $x(t)$ are causal, therefore $\quad y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau \qquad t \geq 0$
- Now, $x(\tau) = e^{-\tau}u(\tau)$ and $h(t-\tau) = e^{-2(t-\tau)}u(t-\tau)$
- And $u(\tau) = 1$ and $u(t-\tau) = 1$
- Therefore

$$y(t) = \int_0^t e^{-\tau}e^{-2(t-\tau)}\,d\tau \qquad t \geq 0$$

- Remember that this integration is with respect to $\tau$ (and not t), $e^{-2t}$ can be pulled outside the integral:

$$y(t) = e^{-2t}\int_0^t e^{\tau}\,d\tau = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t} \qquad t \geq 0$$

- Therefore

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

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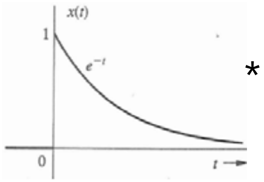
# Example (2)

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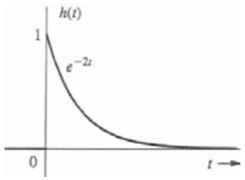
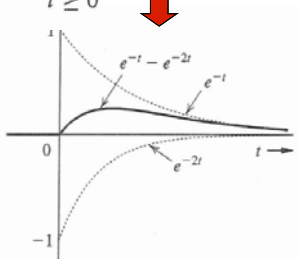



$x(t) = e^{-t}u(t)$

$h(t) = e^{-2t}u(t)$

$$y(t) = \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau \qquad t \geq 0$$




$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_0^t x(\tau)h(t-\tau)d\tau \\ &= \int_0^t e^{-\tau}e^{-2(t-\tau)}d\tau \\ &= (e^{-t} - e^{-2t})u(t) \end{aligned}$$

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$$y(t) = (e^{-t} - e^{-2t})u(t)$$



# Relating this lecture to other courses

- Convolution has been introduced last year in the communication course. We will go deeper into convolution and its physical implication in more details in this lecture.
- Zero-state response (as determined through the convolution operation) is very important, and is intimately related to the zero-input response and the characteristic modes of the system.
- All these are relevant to the 2nd year control course.
- You will also come across convolution again in your 2nd year Communications course and third year DSP course.